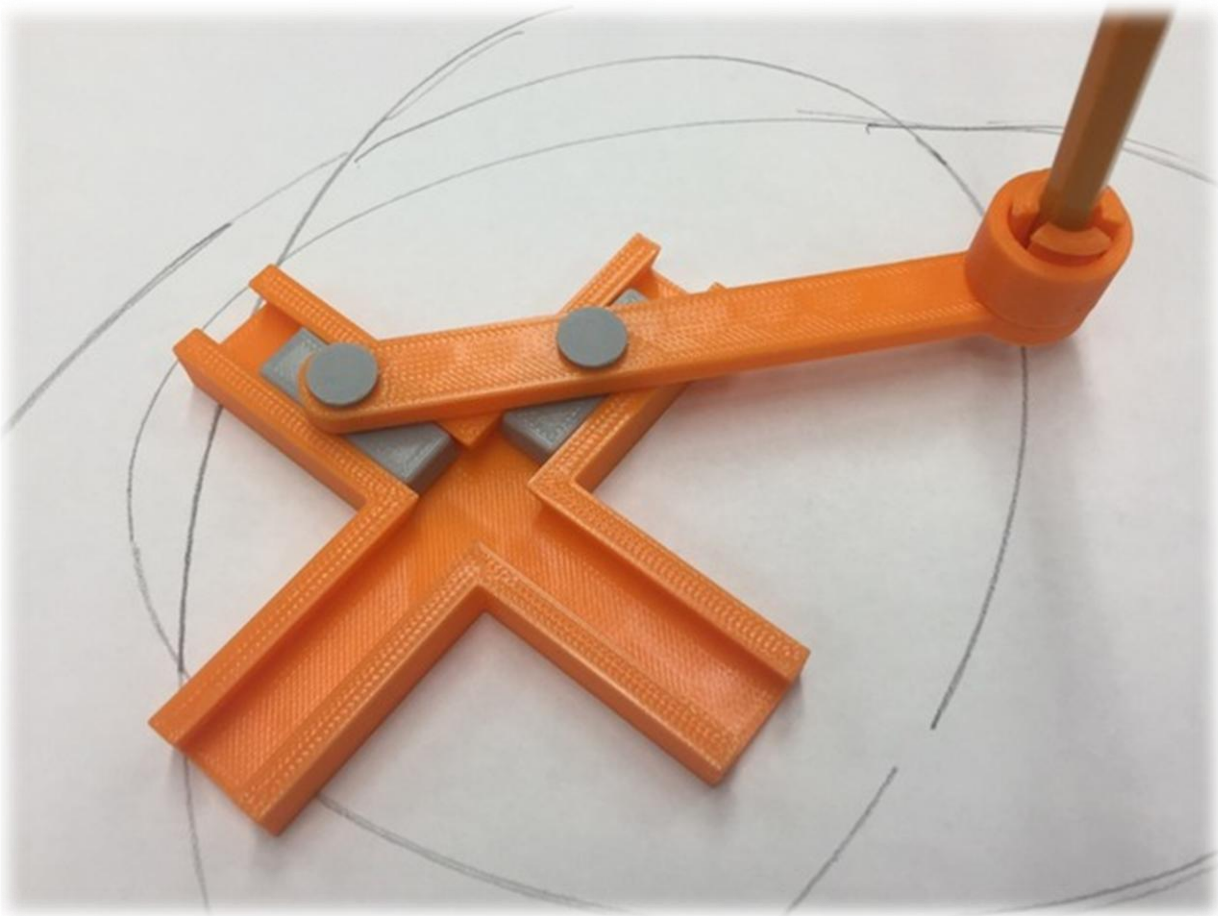
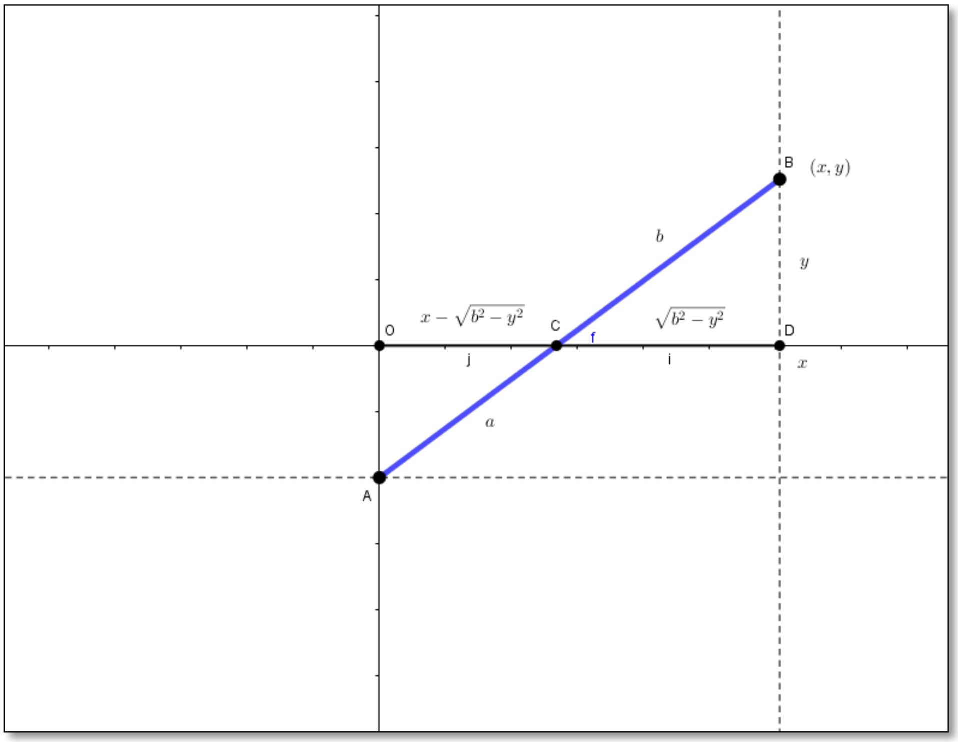
Trammel of Archimedes: From Physical Construction to Proof



Source: Bu, Lingguo. (2017). https://www.thingiverse.com/thing:2695548

Proof One:



In the figure above, *AC* has a fixed length of *a*, *BC* has a fixed length of *b*. Let *B* = (*x*, *y*) be a point on the curve.

Note that

$$CD = \sqrt{b^2 - y^2}, \text{ and } OC = x - \sqrt{b^2 - y^2}.$$

Triangle *ACO* is similar to Triangle *BCD*, which yields

$$\frac{OC}{CD} = \frac{a}{b}, \text{ or, } \frac{x-\sqrt{b^2-y^2}}{\sqrt{b^2-y^2}} = \frac{a}{b}, \text{ which can be simplified as}$$

$$\frac{x}{\sqrt{b^2-y^2}} - 1 = \frac{a}{b}, \text{ and further}$$

$$\frac{x}{\sqrt{b^2-y^2}} = \frac{a+b}{b}.$$

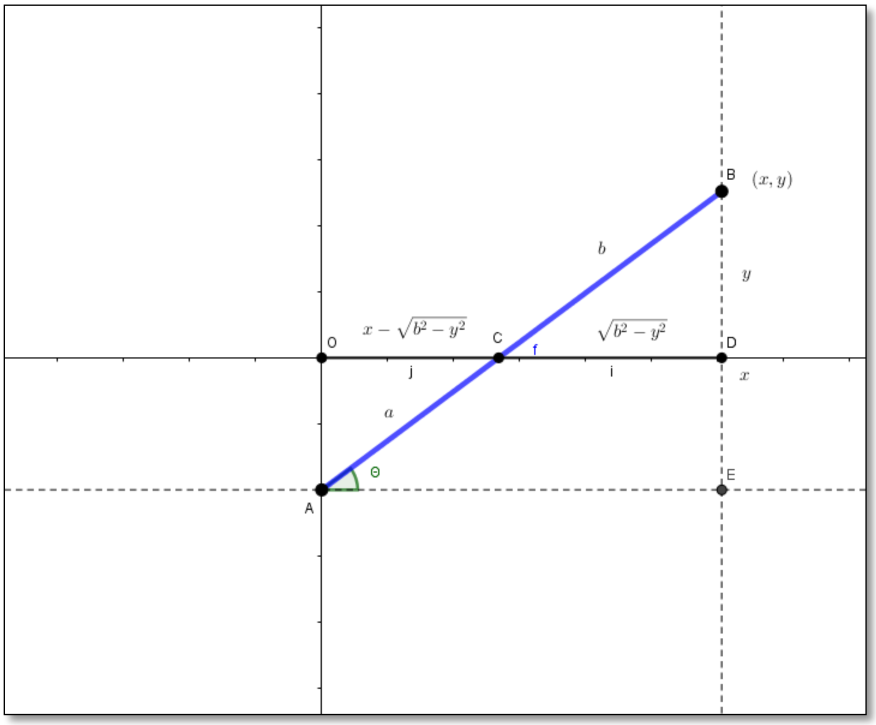
Squaring both side of the above, we get

$$\frac{x^2}{b^2-y^2} = \frac{(a+b)^2}{b^2},$$

$$\frac{x^2}{(a+b)^2} = 1 - \frac{y^2}{b^2}, \text{ and finally}$$

$$\frac{x^2}{(a+b)^2} + \frac{y^2}{b^2} = 1.$$

Proof Two:



In the figure above, *AC* has a fixed length of *a*, *BC* has a fixed length of *b*. Let *B* = *(x, y)* be a point on the curve. Label $\theta = \angle EAB$. Then, using a bit of trig, we get

$$y = b \sin\theta \;,$$

$$x = (a + b) \cos\theta.$$

Then, squaring both equations above, we have

$$y^2 = b^2 \sin^2\theta \;,\; x^2 = (a + b)^2 cos^2\theta.$$

Focusing on the trig functions, we get

$$\frac{y^2}{b^2} = \sin^2\theta \text{ , and } \frac{x^2}{(a+b)^2} = cos^2\theta, \text{ which yield}$$

$$\frac{x^2}{(a+b)^2} + \frac{y^2}{b^2} = sin^2\theta + cos^2\theta = 1.$$